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ON THE CONVERGENCE OF MULTIDIMENSIONAL g-FRACTION

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In this paper we consider the multidimensional g-fraction which is a generalization of the continued g-fraction. We investigate the convergence and also establish estimates of the convergence rate for such fraction in some domains of the space $\mathbb{C}^N$.

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В статье рассматривается многомерная g-дробь, которая является обобщением непрерывной g-дроби. Исследуется сходимость и также получены оценки скорости сходимости для такой дроби в некоторых областях пространства $\mathbb{C}^N$.

1. Introduction. In the analytic theory of continued fractions different form of functional continued fractions are studied [7, 8]. The most studied type is that of g-fractions of the form

$$\frac{s_0}{1} + \frac{g_1 z}{1} + \frac{g_2(1-g_1)z}{1} + \frac{g_3(1-g_2)z}{1} + \cdots, \tag{1}$$

where $s_0 > 0$, $0 < g_n < 1$, $n = \overline{1,\infty}$, $z \in \mathbb{C}$. The fractions of form (1), where $z = 1$, $\lim_{n\to\infty} g_n(1-g_{n-1}) = 0$, were first investigated by Sleszynski [13]. The question of convergence of g-fractions was considered by Van Vleck [15], Perron [10], Scott and Wall [12], Wall [16]. In particular, they proved that continued fraction (1) converges to a function holomorphic in the cut plane $G = \{z \in \mathbb{C} : \ |\arg(1+z)| < \pi\}$, moreover the convergence is uniform on each compact subset of this domain. The estimates of the convergence rate of g-fractions in their domain was established in papers by Merkes [9], Gragg [6]: if $g(z)$ denotes a holomorphic function to which the g-fraction (1) converges in the domain G, then the following estimates hold for the errors of approximation

$$|g(z) - g_n(z)| \le$$
$$\le \max\left[1, \operatorname{tg}\left|\frac{\arg z}{2}\right|\right] \frac{s_0}{\operatorname{Re}\left(\sqrt{1+z}\right)} \left|\sqrt{1+z} - \frac{1}{\sqrt{1+z}}\right| \left|\frac{1-\sqrt{1+z}}{1+\sqrt{1+z}}\right|^{n-1}, \ n = \overline{2,\infty},$$

where $g_n(z)$ is the n-th approximant of continued fraction (1). Different applications of the g-fractions were considered in papers by Wall [16], Thale [14], Runckel [11] and others. In

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particular, the g-fractions were used for analytic continuation of functions, finding of zeros, poles and domains of univalence of some analytic and meromorphic functions, solving of the power moment problem.

2. Definition and notation. The first multidimensional generalization of g-fractions was considered in [1, 2, 4], where the circular domain of convergence for suggested generalization was investigated.

The branched continued fractions (BCF)[1–5] are multidimensional generalizations of continued fractions. Here we give some information concerning of the theory of BCF; see [2] for details.

The expression

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)}}{b_{i(k)}} = b_0 + \sum_{i_1=1}^{N} \cfrac{a_{i(1)}}{b_{i(1)} + \sum\limits_{i_2=1}^{N} \cfrac{a_{i(2)}}{b_{i(2)} + \ddots}}, \tag{2}$$

where $i(k) = i_1, i_2, \dots, i_k$ is short writing of multiindex, $b_0,\ a_{i(k)},\ b_{i(k)}$ are complex numbers, is called a branched continued fraction with N branches of branching.

Fraction (2) is said to converge, if there exists a finite limit of its n-th approximants f_n for $n \to \infty$, where

$$f_n = b_0 + \mathop{\mathrm{D}}_{k=1}^{n} \sum_{i_k=1}^{N} \frac{a_{i(k)}}{b_{i(k)}}.$$

We introduce the short notation for the remainders of BCF: (2)

$$Q_{i(s)}^{(s)} = b_{i(s)}, \quad Q_{i(p)}^{(s)} = b_{i(p)} + \mathop{\mathrm{D}}_{r=p+1}^{s} \sum_{i_r=1}^{N} \frac{a_{i(r)}}{b_{i(r)}}, \tag{3}$$

where $s = \overline{1,\infty},\ p = \overline{1,s-1},\ i_k = \overline{1,N},\ k = \overline{1,s}$. Under this notation the following recurrent relations hold

$$Q_{i(p)}^{(s)} = b_{i(p)} + \sum_{i_{p+1}=1}^{N} \frac{a_{i(p+1)}}{Q_{i(p+1)}^{(s)}},$$

where $s = \overline{1,\infty},\ p = \overline{1,s-1},\ i_k = \overline{1,N},\ k = \overline{1,s-1}$.

Any BCF with f_n^*-th approximants is called the even part of a BCF with f_n-th approximants, if $f_n^* = f_{2n},\ n = \overline{1,\infty}$.

Any BCF with f_n^{**}-th approximants is called the majorant of a BCF with f_n-th approximants, if there exist a natural number n_0 and a positive constant M, such that the relation $|f_n - f_m| \le M|f_n^{**} - f_m^{**}|$ is valid for all $n \ge n_0,\ m \ge n_0$.

A branched continued fraction of the form

$$\frac{s_0}{1} + \sum_{i_1=1}^{N} \frac{g_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{N} \frac{(1-g_{i(1)}) g_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{N} \frac{(1-g_{i(2)}) g_{i(3)} z_{i_3}}{1} + \cdots, \tag{4}$$

where $s_0 > 0,\ 0 < g_{i(k)} < 1,\ k = \overline{1,\infty},\ i_p = \overline{1,N},\ p = \overline{1,k},\ z = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N$, is called a multidimensional g-fraction [2].

In the present paper we investigate the convergence of the multidimensional g-fraction (4) in some unbounded domain of the space $\mathbb{C}^N$ to a holomorphic function $g(z)$. We also establish that BCF (4) converges to a holomorphic function $g(z)$ in some bounded domains, in particular, circular domain, at least as fast as a geometric series with the denominator q, where $0 < q < 1$, $q = q(K)$, K is an arbitrary compact subset of the corresponding bounded domain.

3. The convergence of the multidimensional g-fraction. We consider a BCF of the form

$$\frac{\pi_0}{1+\sum\limits_{i=1}^{N} z_i} - \sum_{i_1=1}^{N} \frac{z_{i_1}}{1} + \frac{\pi_{i(1)}}{1+\sum\limits_{i=1}^{N} z_i} - \sum_{i_2=1}^{N} \frac{z_{i_2}}{1} + \frac{\pi_{i(2)}}{1+\sum\limits_{i=1}^{N} z_i} - \cdots, \tag{5}$$

where all $\pi_{i(k)} > 0$, $z = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N$ and $\sum\limits_{i=1}^{N} z_i \neq -1$, which is called a multidimensional π-fraction, by analogy with the one-dimensional case [7].

By means of the properties of multidimensional linear fractional transformations, the following statement is proved in [3].

Proposition 1. *The even part of a multidimensional π-fraction* (5) *is the multidimensional g-fraction* (4), *where* $s_0 = \pi_0$, $g_{i(k)} = \dfrac{\pi_{i(k)}}{1+\pi_{i(k)}}$, $k = \overline{1,\infty}$, $i_p = \overline{1,N}$, $p = \overline{1,k}$, $z \in \mathbb{C}^N$.

Corollary 1. *The even part of a branched continued fraction*

$$\frac{\pi_0}{1-\sum\limits_{i=1}^{N} |z_i|} + \sum_{i_1=1}^{N} \frac{|z_{i_1}|}{1} + \frac{\pi_{i(1)}}{1-\sum\limits_{i=1}^{N} |z_i|} + \sum_{i_2=1}^{N} \frac{|z_{i_2}|}{1} + \frac{\pi_{i(2)}}{1-\sum\limits_{i=1}^{N} |z_i|} + \cdots, \tag{6}$$

where all $\pi_{i(k)} > 0$, $\sum\limits_{i=1}^{N} |z_i| \neq 1$, *is the BCF of the form*

$$\frac{s_0}{1} - \sum_{i_1=1}^{N} \frac{g_{i(1)}\,|z_{i_1}|}{1} - \sum_{i_2=1}^{N} \frac{g_{i(2)}(1-g_{i(1)})\,|z_{i_2}|}{1} - \sum_{i_3=1}^{N} \frac{g_{i(3)}(1-g_{i(2)})\,|z_{i_3}|}{1} - \cdots, \tag{7}$$

where $s_0 = \pi_0$, $g_{i(k)} = \dfrac{\pi_{i(k)}}{1+\pi_{i(k)}}$, $k = \overline{1,\infty}$, $i_p = \overline{1,N}$, $p = \overline{1,k}$, $z \in \mathbb{C}^N$.

In the following lemma we find the majorant of multidimensional g-fraction (4) in the domain

$$Q = \left\{ z \in \mathbb{C}^N : \quad \sum_{i=1}^{N} |z_i| < 1 \right\}. \tag{8}$$

Lemma 1. *The majorant of multidimensional g-fraction* (4) *in the domain* (8) *is BCF* (7).

Proof. By analogy to (3), we introduce the notation $Q^{(s)}_{i(n)}$, $\tilde{Q}^{(s)}_{i(n)}$, $s=\overline{0,\infty}$, $n=\overline{0,s}$, $i_p=\overline{1,N}$, $p=\overline{1,n}$, $i(0)=0$, $g_0=0$, for the fractions (4) and (7), respectively.

By mathematical induction we verify validity of inequalities

$$\left|Q^{(s)}_{i(n)}\right| \geq \tilde{Q}^{(s)}_{i(n)} > g_{i(n)}, \tag{9}$$

where $s=\overline{0,\infty}$, $n=\overline{0,s}$, $i_p=\overline{1,N}$, $p=\overline{1,n}$, $i(0)=0$, $g_0=0$.

Indeed, relations (9) are obvious for $n=s$. Let inequalities (9) hold for $n=p+1$, where $p+1\leq s$. Then for $n=p$ we obtain

$$\left|Q^{(s)}_{i(p)}\right| \geq 1-\sum_{i_{p+1}=1}^{N}\frac{g_{i(p+1)}(1-g_{i(p)})|z_{i_{p+1}}|}{\left|Q^{(s)}_{i(p+1)}\right|} \geq 1-\sum_{i_{p+1}=1}^{N}\frac{g_{i(p+1)}(1-g_{i(p)})|z_{i_{p+1}}|}{\tilde{Q}^{(s)}_{i(p+1)}} = \tilde{Q}^{(s)}_{i(p)}.$$

By virtue of estimates (9), $\tilde{Q}^{(s)}_{i(p+1)}>0$. Therefore, replacing $g_{i(p+1)}$ by $\tilde{Q}^{(s)}_{i(p+1)}$, inequalities (9) are obtained for $n=p$.

It follows from inequalities (9) that $Q^{(s)}_{i(n)}\neq 0$ and $\tilde{Q}^{(s)}_{i(n)}>0$ for all indices. Let $g_n(z), \tilde{g}_n(z)$ be the n-th approximants of BCFs (4) and (7), respectively. Applying the well-known formula for the difference of two approximants of BCF, see [2], for fraction (4) we obtain

$$|g_n(z)-g_m(z)| \leq \sum_{i_1,i_2,\dots,i_m=1}^{N}\frac{s_0\prod\limits_{r=1}^{m} g_{i(r)}(1-g_{i(r-1)})|z_{i_r}|}{\prod\limits_{r=0}^{m}\left|Q^{(n-1)}_{i(r)}\right|\prod\limits_{r=0}^{m-1}\left|Q^{(m-1)}_{i(r)}\right|} \leq$$

$$\leq \sum_{i_1,i_2,\dots,i_m=1}^{N}\frac{s_0\prod\limits_{r=1}^{m} g_{i(r)}(1-g_{i(r-1)})|z_{i_r}|}{\prod\limits_{r=0}^{m}\tilde{Q}^{(n-1)}_{i(r)}\prod\limits_{r=0}^{m-1}\tilde{Q}^{(m-1)}_{i(r)}} = \tilde{g}_n(z)-\tilde{g}_m(z),$$

where $n>m\geq 1$. From this there follows the statement of lemma. □

The following theorem contains estimates of the convergence rate for multidimensional g-fraction (4) in the circular domain (8).

Theorem 1. *The multidimensional g-fraction (4) converges to a function $g(z)$ holomorphic in the domain (8). The following estimates of the truncation error hold in the domain Q:*

$$|g(z)-g_k(z)| \leq \frac{s_0\left(\sum\limits_{i=1}^{N}|z_i|\right)^k}{\left(1-T\sum\limits_{i=1}^{N}|z_i|\right)\left(1+\nu_{k-1}\left(1-\sum\limits_{i=1}^{N}|z_i|\right)\right)}\prod_{p=1}^{k}\frac{\mu_p}{\mu_p+1-T\sum\limits_{i=1}^{N}|z_i|}\times$$

$$\times\prod_{p=1}^{k-2}\frac{2+\nu_{p+1}\left(1-\sum\limits_{i=1}^{N}|z_i|\right)}{2+(\nu_{p+1}+\nu_p(2+\nu_{p+1}))\left(1-\sum\limits_{i=1}^{N}|z_i|\right)} \leq \frac{s_0\left(\sum\limits_{i=1}^{N}|z_i|\right)^k}{1-T\sum\limits_{i=1}^{N}|z_i|}, \quad k=\overline{2,\infty}, \tag{10}$$

where $g_k(z)$ is the n-th approximant of fraction (4), T, μ_p, ν_p are defined in the following way:

$$T = \sup_{n\in\mathbb{N}} \left\{ 1 - \left(1 + \sum_{r=n}^{\infty} \mu_n \mu_{n+1} \dots \mu_r \right)^{-1} \right\} \le 1, \tag{11}$$
$$\mu_p = \max\left\{ g_{i(p)}(1-g_{i(p)})^{-1},\ i_r = \overline{1,N},\ r = \overline{1,p} \right\},$$
$$\nu_p = \min\left\{ g_{i(p)}(1-g_{i(p)})^{-1},\ i_r = \overline{1,N},\ r = \overline{1,p} \right\}.$$

Proof. According to Lemma 1 and Corollary 1 for $k \ge 1$, $m \ge 1$ we obtain

$$|g_{k+m}(z) - g_k(z)| \le \tilde{g}_{k+m}(z) - \tilde{g}_k(z) = \tilde{f}_{2k+2m}(z) - \tilde{f}_{2k}(z),$$

where $\tilde{g}_n(z)$, $\tilde{f}_n(z)$ are the n-th approximants of BCFs (7) and (6), respectively. Let us estimate the difference between two approximants $\tilde{f}_{2k+2m}(z) - \tilde{f}_{2k}(z)$ of the branched continued fraction (6).

Introduce the notation for the remainders of BCF (6)

$$\tilde{F}^{(s)}_{i(s)} = 1 - \sum_{i=1}^{N} |z_i|, \quad \tilde{F}^{(s)}_{i(p)} = 1 - \sum_{i=1}^{N} |z_i| +$$
$$+ \sum_{i_{p+1}=1}^{N} \frac{|z_{i_{p+1}}|}{1} + \frac{\pi_{i(p+1)}}{1 - \sum\limits_{i=1}^{N} |z_i|} + \dots + \sum_{i_{s-1}=1}^{N} \frac{|z_{i_{s-1}}|}{1} + \frac{\pi_{i(s-1)}}{1 - \sum\limits_{i=1}^{N} |z_i|} + \sum_{i_s=1}^{N} \frac{|z_{i_s}|}{1}, \tag{12}$$

where $s = \overline{0,\infty}$, $p = \overline{0,s-1}$, $i_k = \overline{1,N}$, $k = \overline{1,s}$, $i(0) = 0$,

$$\tilde{G}^{(s)}_{i(s)} = 1, \quad \tilde{G}^{(s)}_{i(p)} = 1 + \frac{\pi_{i(p)}}{1 - \sum\limits_{i=1}^{N} |z_i|} + \sum_{i_{p+1}=1}^{N} \frac{|z_{i_{p+1}}|}{1} + \dots + \frac{\pi_{i(s-1)}}{1 - \sum\limits_{i=1}^{N} |z_i|} + \sum_{i_s=1}^{N} \frac{|z_{i_s}|}{1}, \tag{13}$$

where $s = \overline{1,\infty}$, $p = \overline{1,s-1}$, $i_k = \overline{1,N}$, $k = \overline{1,s}$.

Under this notation the following recurrent relations hold

$$\tilde{F}^{(s)}_{i(p)} = 1 - \sum_{i=1}^{N} |z_i| + \sum_{i_{p+1}=1}^{N} \frac{|z_{i_{p+1}}|}{\tilde{G}^{(s)}_{i(p+1)}},$$

where $s = \overline{0,\infty}$, $p = \overline{0,s-1}$, $i_k = \overline{1,N}$, $k = \overline{1,s-1}$, $i(0) = 0$,

$$\tilde{G}^{(s)}_{i(p)} = 1 + \frac{\pi_{i(p)}}{\tilde{F}^{(s)}_{i(p)}}, \tag{14}$$

where $s = \overline{1,\infty}$, $p = \overline{1,s-1}$, $i_k = \overline{1,N}$, $k = \overline{1,s-1}$.

By mathematical induction we verify validity of inequalities

$$\tilde{F}^{(r)}_{i(n)} \ge D^{r-1}_{n+1}(\mu, z) > 0, \quad r = \overline{1,\infty},\ n = \overline{0,r-1},\ i_p = \overline{1,N},\ p = \overline{1,n},\ i(0) = 0, \tag{15}$$

where

$$D_n^m(\mu,z)=1+\mathop{\mathrm{D}}_{r=n}^{m}\frac{-\mu_r\sum\limits_{i=1}^{N}|z_i|}{1+\mu_r},\quad \text{if}\quad n\le m,\quad D_n^m(\mu,z)=1,\quad \text{if}\quad n>m.$$

For $n=r-1$ we obtain $\tilde F_{i(r-1)}^{(r)}=1=D_r^{r-1}(\mu,z)$. Suppose that relations (15) hold for $n=p+1$, where $p+1\le r-1$. Then for $n=p$ we obtain

$$\tilde F_{i(p)}^{(r)}=1-\sum_{i_{p+1}=1}^{N}\frac{|z_{i_{p+1}}|\pi_{i(p+1)}}{\pi_{i(p+1)}+\tilde F_{i(p+1)}^{(r)}}\ge 1-\frac{\mu_{p+1}}{\mu_{p+1}+D_{p+2}^{r-1}(\mu,z)}\sum_{i=1}^{N}|z_i|\,.$$

The last expression is positive and equal to $D_{p+1}^{r-1}(\mu,z)$. Inequalities (15) are proved.

By analogy, we prove validity of the following inequalities

$$\tilde F_{i(n)}^{(r)}\le D_{n+1}^{r-1}(\nu,z),\quad r=\overline{1,\infty},\ n=\overline{0,r-1},\ i_p=\overline{1,N},\ p=\overline{1,n},\ i(0)=0,$$

where

$$D_n^m(\nu,z)=1+\mathop{\mathrm{D}}_{r=n}^{m}\frac{-\nu_r\sum\limits_{i=1}^{N}|z_i|}{1+\nu_r},\quad \text{if}\quad n\le m, D_n^m(\nu,z)=1,\quad \text{if}\quad n>m.$$

Applying the latter inequality, one can show that

$$\tilde G_{i(n)}^{(r)}\ge 1+\nu_n\left(D_{n+1}^{r-1}(\nu,z)\right)^{-1}>0,\quad r=\overline{1,\infty},\ n=\overline{1,r-1},\ i_p=\overline{1,N},\ p=\overline{1,n}. \tag{16}$$

From relations (14)–(16) it follows that $\tilde F_{i(n)}^{(r)}>0$, $\tilde G_{i(n)}^{(r)}>0$ for all indices. Applying the method suggested in the monograph [2] and relations (14), we find a formula for the difference of two approximants of BCF (6) for $k\ge 1$, $m\ge 1$, namely

$$\tilde f_{2k+2m}(z)-\tilde f_{2k}(z)=\frac{\pi_0}{\tilde F_0^{(k+m)}}\sum_{i_1,i_2,\dots,i_k=1}^{N}\frac{\prod\limits_{r=1}^{k}|z_{i_r}|\pi_{i(r)}}{\prod\limits_{r=1}^{k}\left(\pi_{i(r)}+\tilde F_{i(r)}^{(k+m)}\right)\prod\limits_{r=1}^{k-1}\tilde F_{i(r-1)}^{(k)}\tilde G_{i(r)}^{(k)}}.$$

Let us estimate the following expressions

$$L_{i(p-1)}=\sum_{i_p=1}^{N}\frac{\left|z_{i_p}\right|\pi_{i(p)}}{\left(\pi_{i(p)}+\tilde F_{i(p)}^{(k+m)}\right)\tilde F_{i(p-1)}^{(k)}\tilde G_{i(p)}^{(k)}},$$

where $p=\overline{1,k-1}$, $i_r=\overline{1,N}$, $r=\overline{1,p}$, $i(0)=0$. Using relations (14)–(16), for an arbitrary

multiindex $i(p-1)$ we obtain

$$L_{i(p-1)} \le \frac{\mu_p \sum\limits_{i_p=1}^{N} \frac{|z_{i_p}|}{\tilde{G}^{(k)}_{i(p)}}}{\left(\mu_p + D^{k+m-1}_{p+1}(\mu,z)\right)\left(1 - \sum\limits_{i=1}^{N}|z_i| + \sum\limits_{i_p=1}^{N}\frac{|z_{i_p}|}{\tilde{G}^{(k)}_{i(p)}}\right)} \le$$

$$\le \frac{\mu_p \sum\limits_{i=1}^{N}|z_i|}{\left(\mu_p + D^{k+m-1}_{p+1}(\mu,z)\right)\left(\sum\limits_{i=1}^{N}|z_i| + \left(1 - \sum\limits_{i=1}^{N}|z_i|\right)\left(1 + \frac{\nu_p}{D^{k-1}_{p+1}(\nu,z)}\right)\right)}.$$

Applying the latter inequality, we obtain

$$\tilde{f}_{2k+2m}(z) - \tilde{f}_{2k}(z)$$
$$\le \frac{\pi_0}{D^{k+m-1}_1(\mu,z)}\left(\sum_{i=1}^{N}|z_i|\right)^k \prod_{p=1}^{k}\frac{\mu_p}{\mu_p + D^{k+m-1}_{p+1}(\mu,z)}\prod_{p=1}^{k-1}\frac{1}{1 + \frac{\nu_p}{D^{k-1}_{p+1}(\nu,z)}\left(1 - \sum\limits_{i=1}^{N}|z_i|\right)}. \quad (17)$$

By equivalent transformations we reduce the fraction $D^{\infty}_1(\mu,z)$ to the form

$$1 + \mathop{\mathrm{D}}_{r=1}^{\infty}\frac{q_r(q_{r-1}-1)\sum\limits_{i=1}^{N}|z_i|}{1}, \quad (18)$$

where $q_0 = 0$, $q_r = \mu_r(1+\mu_r)^{-1}$, $r = \overline{1,\infty}$. Applying the equivalent transformations and Theorem 11.1 [16], we can prove the following inequalities

$$D^{k-1}_{p+1}(\nu,z) \le 1 - \frac{\nu_{p+1}}{2+\nu_{p+1}}\sum_{i=1}^{N}|z_i|, \quad k = \overline{3,\infty},\ p = \overline{1,k-2}.$$

With the proofs of Theorems 11.1–11.3 [16] it implies that continued fraction (18) converges unconditionally and its sequence of approximants is a monotonically decreasing. The value of fraction (18) is not less than $1 - T\sum\limits_{i=1}^{N}|z_i|$. Therefore, passing to the limit in inequalities (17) as $m \to \infty$ and by computations, we obtain estimates (10). □

Theorem 2. *The multidimensional g-fraction* (4) *converges to a function* $g(z)$ *holomorphic in the domain*

$$D = \bigcup_{\alpha\in(-\pi/2,\pi/2)}(Q_\alpha \cap P_\alpha), \quad (19)$$

where

$$Q_\alpha = \left\{ z \in \mathbb{C}^N : \quad \sum_{n=1}^{N} |z_n| < 4\cos^2\alpha - 2T \sum_{n=1}^{N} \left(|z_n| - \operatorname{Re}\left(z_n e^{-2i\alpha} \right) \right) \right\},$$

$$P_\alpha = \left\{ z \in \mathbb{C}^N : \quad \sum_{n=1}^{N} \left(|z_n| - \operatorname{Re}\left(z_n e^{-2i\alpha} \right) \right) < 2\cos^2\alpha \right\}, \tag{20}$$

T defined by (11). The following estimates of the truncation error hold in the domain $Q_\alpha \bigcap P_\alpha$ for every α, $-\pi/2 < \alpha < \pi/2$,

$$|g(z) - g_k(z)| \le \frac{16 s_0 \mu_k \cos\alpha}{\mu_k + 1 - \sum\limits_{n=1}^{N} \dfrac{|z_n| - \operatorname{Re}\left(z_n e^{-2i\alpha} \right)}{2\cos^2\alpha}} \times$$

$$\times \frac{\left(\sum\limits_{i=1}^{N} |z_i| \right)^k}{\left(4\cos^2\alpha - 2T \sum\limits_{n=1}^{N} \left(|z_n| - \operatorname{Re}\left(z_n e^{-2i\alpha} \right) \right) \right)^{k+1}}, \quad k = \overline{2, \infty}, \tag{21}$$

where $g_k(z)$ is the n-th approximant of BCF (4).

Proof. Let α be an arbitrary number in the interval $(-\pi/2, \pi/2)$. Let us estimate the modulus of the difference of approximants $|f_{2k+2m}(z) - f_{2k}(z)|$ for BCF (5) and $k \ge 1$, $m \ge 1$.

By analogy to (12) and (13), we introduce the notation $F^{(s)}_{i(n)}$, $G^{(s+1)}_{i(n+1)}$, $s = \overline{0, \infty}$, $n = \overline{0, s}$, $i_k = \overline{1, N}$, $k = \overline{1, s}$, $i(0) = 0$, for the remainders of BCF (5). Under this notation the following recurrent relations hold for multidimensional π-fraction (5)

$$F^{(s)}_{i(s)} = 1 + \sum_{i=1}^{N} z_i, \quad F^{(s)}_{i(p)} = 1 + \sum_{i=1}^{N} z_i - \sum_{i_{p+1}=1}^{N} \frac{z_{i_{p+1}}}{G^{(s)}_{i(p+1)}}, \tag{22}$$

where $s = \overline{0, \infty}$, $p = \overline{0, s-1}$, $i_k = \overline{1, N}$, $k = \overline{1, s}$, $i(0) = 0$,

$$G^{(s)}_{i(s)} = 1, \quad G^{(s)}_{i(p)} = 1 + \frac{\pi_{i(p)}}{F^{(s-1)}_{i(p)}}, \tag{23}$$

where $s = \overline{1, \infty}$, $p = \overline{1, s-1}$, $i_k = \overline{1, N}$, $k = \overline{1, s}$.

Using relations (22) and (23), by mathematical induction we show that the following inequalities are valid

$$\operatorname{Re}\left(F^{(r)}_{i(n)} e^{-i\alpha} \right) \ge D^{r-1}_{n+1}(q, w) \cos\alpha > 0, \quad r = \overline{1, \infty},\ n = \overline{0, r-1},\ i_p = \overline{1, N},\ p = \overline{1, n}, \tag{24}$$

where

$$D^m_n(q, w) = 1 + q_n w \left(1 + \mathop{\mathrm{D}}_{r=n+1}^{m} \frac{q_r(1 - q_{r-1})w}{1} \right)^{-1}, \quad \text{if} \quad n < m, \quad D^n_n(q, w) = 1 + q_n w,$$

$$D_n^m(q,w)=1, \quad \text{if} \quad n>m, \quad w=\sum_{n=1}^{N}\frac{\operatorname{Re}\left(z_n e^{-2i\alpha}\right)-|z_n|}{2\cos^2\alpha}, \quad q_r=\frac{\mu_r}{1+\mu_r}.$$

For $n=r-1$ relations (24) are obvious. By induction hypothesis, (24) hold for $n=p+1$, where $p+1\le r-1$, and we prove (24) for $n=p$. Indeed, relations (22) and (23) readily imply

$$F_{i(p)}^{(r)}e^{-i\alpha}=\left(1+\sum_{i=1}^{N}z_i\right)e^{-i\alpha}-\sum_{i_{p+1}=1}^{N}\frac{z_{i_{p+1}}e^{-i\alpha}}{G_{i(p+1)}^{(r)}}=e^{-i\alpha}+\sum_{i_{p+1}=1}^{N}\frac{\pi_{i(p+1)}z_{i_{p+1}}e^{-2i\alpha}}{\left(\pi_{i(p+1)}+F_{i(p+1)}^{(r)}\right)e^{-i\alpha}}.$$

In the proof of Lemma 4.41 [7] it was shown that for $x\ge c>0$ and $v^2\le 4u+4$

$$\min_{-\infty<y<\infty}\operatorname{Re}\frac{u+iv}{x+iy}=-\frac{\sqrt{u^2+v^2}-u}{2x}. \tag{25}$$

Using relation (25), where

$$u=\operatorname{Re}\left(\pi_{i(p+1)}z_{i_{p+1}}e^{-2i\alpha}\right), \quad v=\operatorname{Im}\left(\pi_{i(p+1)}z_{i_{p+1}}e^{-2i\alpha}\right),$$
$$x=\operatorname{Re}\left(\left(\pi_{i(p+1)}+F_{i(p+1)}^{(r)}\right)e^{-i\alpha}\right), \quad y=\operatorname{Im}\left(\left(\pi_{i(p+1)}+F_{i(p+1)}^{(r)}\right)e^{-i\alpha}\right),$$

and the induction hypothesis, we obtain

$$\operatorname{Re}\left(F_{i(p)}^{(r)}e^{-i\alpha}\right)\ge\cos\alpha-\sum_{i_{p+1}=1}^{N}\frac{\pi_{i(p+1)}\left(\left|z_{i_{p+1}}\right|-\operatorname{Re}\left(z_{i_{p+1}}e^{-2i\alpha}\right)\right)}{2\left(\pi_{i(p+1)}\cos\alpha+\operatorname{Re}\left(F_{i(p+1)}^{(r)}e^{-i\alpha}\right)\right)}\ge$$
$$\ge\left(1+\frac{\mu_{p+1}w}{\mu_{p+1}+D_{p+2}^{r-1}(q,w)}\right)\cos\alpha=D_{p+1}^{r-1}(q,w)\cos\alpha>0.$$

Inequalities (24) are proved.

From relations (22)–(24) it follows that $F_{i(n)}^{(r)}\ne 0$, $G_{i(n)}^{(r)}\ne 0$ for all indices. Applying the method suggested in [2] and relations (22), we find a formula for the difference of two approximants of BCF (5) for $k\ge 1$, $m\ge 1$, namely

$$f_{2k+2m}(z)-f_{2k}(z)=\frac{\pi_0}{F_0^{(k+m)}F_0^{(k)}}\sum_{i_1,i_2,\dots,i_k=1}^{N}\frac{\prod\limits_{r=1}^{k}z_{i_r}\pi_{i(r)}}{\prod\limits_{r=1}^{k}\left(\pi_{i(r)}+F_{i(r)}^{(k+m)}\right)\prod\limits_{r=1}^{k-1}\left(\pi_{i(r)}+F_{i(r)}^{(k)}\right)}.$$

We should remark that for $x>0$, $a_1>0$, $a_2>0$

$$\max_x\frac{x}{(x+a_1)(x+a_2)}=\frac{1}{\left(\sqrt{a_1}+\sqrt{a_2}\right)^2}. \tag{26}$$

Taking into account relations (23), (24), (26), for $k\ge 1$, $m\ge 1$ we obtain

$$|f_{2k+2m}(z)-f_{2k}(z)|\le\frac{\pi_0\mu_k}{D_1^{k+m-1}(q,w)D_1^{k-1}(q,w)\left(\mu_k+D_{k+1}^{k+m-1}(q,w)\right)\cos^3\alpha}\times$$
$$\times\left(\sum_{i=1}^{N}|z_i|\right)^k\prod_{n=2}^{k}\frac{1}{\left(\sqrt{D_n^{k+m-1}(q,w)}+\sqrt{D_n^{k-1}(q,w)}\right)^2\cos^2\alpha}. \tag{27}$$

Both the fraction $D_1^\infty(q,w)$ and continued fraction (18) converge unconditionally and their sequence of approximants is a monotonically decreasing. The value of the continued fraction $D_1^\infty(q,w)$ is not less than $1+Tw$. According to Proposition 1 the even part of a multidimensional π-fraction (5) is a multidimensional g-fraction (4). Therefore, passing to the limit in inequality (27) as $m\to\infty$ and by computations, we obtain estimates (21). By definition of Q_α, we conclude that multidimensional g-fraction (4) converges in the domain $Q_\alpha\cap P_\alpha$. By virtue of arbitrarity of α BCF (4) converges in (19). □

Remark. It follows from Theorems 1 and 2 that the modulus of the difference between the values of multidimensional g-fraction (4) and its n-th approximants does not exceed q^n, where $0<q<1$, $q=q(K)$, K is an arbitrary compact subset of bounded domain which has been considered in the corresponding theorem.

The following theorem is the main result of the paper.

Theorem 3. *Multidimensional g-fraction* (4) *converges to a function holomorphic in the domain*

$$P=\bigcup_{\alpha\in(-\pi/2,\pi/2)} P_\alpha, \tag{28}$$

where P_α is defined by (20), *moreover the convergence is uniform on each compact subset of this domain.*

Proof. Since, according to Proposition 1, BCF (4) is the even part of a multidimensional π-fraction (5), we write the approximant $g_n(z)$ in the form

$$g_n(z)=f_{2n}(z)=\frac{s_0}{F_0^{(n)}}.$$

Inequalities (24) imply that the approximants $g_n(z)$, $n=\overline{1,\infty}$, of multidimensional g-fraction (4) form the sequence of functions holomorphic in domain (28).

Let α be an arbitrary number in the interval $(-\pi/2,\pi/2)$, $P_{\alpha,C}$ be a domain which is contained in P_α

$$P_{\alpha,C}=\left\{z\in\mathbb{C}^N:\quad \sum_{n=1}^{N}\left(|z_n|-\operatorname{Re}\left(z_n e^{-2i\alpha}\right)\right)<2C\cos^2\alpha\right\},\quad 0<C<1. \tag{29}$$

Applying each inequality $|D_n^m(q,w)|>1-CT$ (see the proofs of Theorems 11.1-11.3 [16]) and inequalities (24), for arbitrary $z\in P_{\alpha,C}$ and $g_n(z)$, $n\geq 1$, we obtain

$$|g_n(z)|\leq\frac{s_0}{(1-CT)\cos\alpha}=M(P_{\alpha,C}),$$

where the constant $M(P_{\alpha,C})$ depends only on the domain $P_{\alpha,C}$, i. e. the sequence $\{g_n(z)\}$ is uniformly bounded in the domain of form (29).

Let K be an arbitrary compact subset of domain (28). Cover K by the domain of the form (29). Pick a finite subcovering $\left\{P_{\alpha_j,C_j}\right\}_{j=1}^{s}$ of this covering. Let

$$M(K)=\max\{M(P_{\alpha_j,C_j}):\quad j=\overline{1,s}\}.$$

Then for arbitrary $z \in K$ and $g_n(z)$, $n \geq 1$, we obtain $|g_n(z)| \leq M(K)$, i. e. the sequence $\{g_n(z)\}$ is uniformly bounded on each compact subset of domain (28).

According to Theorem 1 fraction (4) converges in the domain

$$\Delta_r = \left\{ z \in \mathbb{C}^N : \quad \sum_{i=1}^{N} |z_i| < r < 1 \right\}.$$

Evidently $\Delta_r \subset P$ for each $0 < r < 1$, in particular, say $\Delta_{1/2} \subset P$. Applying Theorem 2.17 [2], we conclude that the multidimensional g-fraction (4) converges uniformly on each compact subset of domain (28). □

Remark. The authors do not know whether the obtained results are sharp.

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